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Metric characterizations of α-well-posedness for symmetric quasi-equilibrium problems

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Abstract The purpose of this paper is to generalize the concept of α-well-posedness to the symmetric quasi-equilibrium problem. We establish some metric characterizations of α-well-posedness for the symmetric quasi-equilibrium problem. Under some suitable conditions, we prove that the α-well-posedness is equivalent to the existence and uniqueness of solution for the symmetric quasi-equilibrium problems. The corresponding concept of α-well-posedness in the generalized sense is also investigated for the symmetric quasi-equilibrium problem having more than one solution. The results presented in this paper generalize and improve some known results in the literature.

Keywords Symmetric quasi-equilibrium problem · Set-valued mapping · α-well-posedness · α-approximating sequence · Metric characterization

1 Introduction

The notion of well-posedness for an optimization problem was first introduced by Tykhonov [36], already known as Tykhonov well-posedness, which means the existence and uniqueness of solution, and the convergence of every minimizing sequence toward the unique solution. However, in many practical situations, the solution may not be unique for an optimization problem. Thus, the concept of well-posedness in the generalized sense was introduced, which means the existence of solutions and the convergence of some subsequence of ever minimizing sequence toward a solution. In the following years, well-posedness has received much attention due to it plays an important role in the stability theory for optimization problems. A large number of results about well-posedness have appeared in the literature, see,

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e.g., [2,6,8,12,14–17,32,38,39], where Refs. [2,8,12,15,38,39] are for the class of scalar optimization problems, Refs. [6,14,16,32] for the class of vector optimization problems.

In recent years, the concept of well-posedness has been generalized to variational inequality problems [5,4,7,11,17,18,23–25,28], Nash equilibrium problems [25,26,29,30,33,37], inclusion problems [4,11,21], and fixed point problems [4,11,13,21,35]. In particular, Lucchetti and Patrone [28] first introduced the notion of well-posedness for a variational inequality by using Ekeland's variational principle. Lignola and Morgan [24] introduced the parametric well-posedness for a family of variational inequalities. Lignola [23] further introduced the notions of well-posedness and L-well-posedness for quasivariational inequalities and derived some metric characterizations of well-posedness. At the same time, Del Prete et al. [7] introduced the concept of α-well-posedness for a class of variational inequalities. Recently, Fang et al. [11] generalized the concept of well-posedness to a class of mixed variational inequalities in Hilbert spaces. They obtained some metric characterizations of its well-posedness and established the links with the well-posedness of inclusion problems and fixed point problems. Very recently, Ceng and Yao [4] generalized the results of Fang et al. [11] to a class of generalized mixed variational inequalities in Hilbert spaces. Ceng et al. [5] studied the well-posedness for a class of mixed quasivariational-like inequalities in Banach spaces. For the well-posedness of variational inequalities with functional constraints, we refer to Huang and Yang [17] and Huang et al. [18]. On the other hand, in 2006, Lignola and Morgan [26] presented the notion of α-well-posedness for the Nash equilibria problem and gave some metric characterizations of this type well-posedness. Petrusel et al. [35] and Fuster et al. [13] discussed the well-posedness of fixed point problems for multivalued mappings in metric spaces.

It is well known that the equilibrium problem provides a general mathematical model for a wide range of practical problems, which includes as special cases optimization problems, Nash equilibria problems, fixed point problems, variational inequality problems and complementarity problems (see, e.g., [3,22]), and has been investigated intensively. It is worth mentioning that one can equivalently transform a equilibrium problem into a minimizing problem by using gap function and some numerical methods have been developed to solve the equilibrium problem (see, e.g., [31]). This fact motivates researchers to study the well-posedness for equilibrium problems. Recently, Fang et al. [10] introduced the concepts of parametric well-posedness for equilibrium problems and derived some metric characterizations for these types of well-posedness. For the well-posedness of equilibrium problems with functional constraints, we refer readers to [27]. However, to the best of our knowledge, there are no results concerned with the problems of the well-posedness for symmetric quasi-equilibrium problems in Banach spaces.

Motivated and inspired by the works mentioned above, in this paper, we generalize the concept of α-well-posedness to symmetric quasi-equilibrium problems which includes equilibrium problems, Nash equilibrium problems, quasivariational inequalities, variational inequalities and fixed point problems as special cases. Some metric characterizations of α-well-posedness for symmetric quasi-equilibrium problems are given under some suitable conditions. Moreover, we give some examples to illustrate our results. The results presented in this paper generalize and improve some known results due to Ceng et al. [5], Ceng and Yao [4], Fang et al. [10], Fang et al. [11], Lignola [23], and Lignola and Morgan [26].

2 Preliminaries

Throughout this paper, unless specified otherwise, let X and Y be two real Banach spaces, and let $C \subseteq X$ and $D \subseteq Y$ be two nonempty closed and convex subsets. Let $S : C \times D \to 2^C$

and $T : C \times D \to 2^D$ be two set-valued mappings and let $f, g : C \times D \to R$ be two real functions. Suppose that α is a non-negative real number and $N = \{1, 2, \ldots\}$.

In this paper, we consider the following symmetric quasi-equilibrium problem (in short, SQEP) consists in finding a point $(x_0, y_0) \in C \times D$ such that,

$$x_0 \in S(x_0, y_0) \text{ and } f(x_0, y_0) \le f(z, y_0), \quad \forall z \in S(x_0, y_0),$$
$$y_0 \in T(x_0, y_0) \text{ and } g(x_0, y_0) \le g(x_0, w), \quad \forall w \in T(x_0, y_0).$$

This problem was first considered by Noor and Oettli [34], which includes equilibrium problems [3], Nash equilibrium problems [9], quasivariational inequalities [1], variational inequalities [19] and fixed point problems [13,35] as special cases.

It is worth mentioning that Noor and Oettli [34] only established the existence of solutions for (SQEP). Our aim in this paper is to investigate the α-well-posedness for (SQEP) in Banach spaces.

Denote by Γ the solution set of (SQEP). In the sequel we will introduce the notions of α-approximating sequence and of α-well-posedness for (SQEP).

Definition 2.1 A sequence $\{(x_n, y_n)\} \subset C \times D$ is called an α-approximating sequence for (SQEP) iff, there exists a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that,

$$d(x_n, S(x_n, y_n)) \le \varepsilon_n, \text{ i.e., } x_n \in B(S(x_n, y_n), \varepsilon_n), \quad \forall n \in N,$$
$$d(y_n, T(x_n, y_n)) \le \varepsilon_n, \text{ i.e., } y_n \in B(T(x_n, y_n), \varepsilon_n), \quad \forall n \in N,$$

and

$$f(x_n, y_n) - f(z, y_n) \le \varepsilon_n + \frac{\alpha}{2}\|x_n - z\|^2, \quad \forall z \in S(x_n, y_n), \quad \forall n \in N,$$
$$g(x_n, y_n) - g(x_n, w) \le \varepsilon_n + \frac{\alpha}{2}\|y_n - w\|^2, \quad \forall w \in T(x_n, y_n), \quad \forall n \in N,$$

where $B(S(x, y), \varepsilon)$ denotes the ball of radius ε around $S(x, y)$, that is, the set $\{m \in X : d(S(x, y), m) = \inf_{b \in S(x,y)} \|m - b\| \le \varepsilon\}$. When $\alpha = 0$, we say that the sequence $\{(x_n, y_n)\}$ is an approximating sequence for (SQEP).

Definition 2.2 SQEP is said to be α-well-posed if it has a unique solution (x_0, y_0) and for every α-approximating sequence $\{(x_n, y_n)\}$ strongly converges to (x_0, y_0). When $\alpha = 0$, we say that (SQEP) is well-posed.

Definition 2.3 SQEP is said to be α-well-posed in the generalized sense if the solution set Γ of (SQEP) is nonempty and for every α-approximating sequence $\{(x_n, y_n)\}$ has a subsequence which strongly converges to some point of Γ. When $\alpha = 0$, we say that (SQEP) is well-posed in the generalized sense.

In order to investigate the α-well-posedness for (SQEP), we need the following definitions.

Definition 2.4 [20] The Painlevé–Kuratowski limits of a sequence $\{H_n\} \subseteq X$ are defined by

$$\liminf_n H_n = \{y \in X : \exists y_n \in H_n, n \in N, \text{ with } \lim_n y_n = y\},$$
$$\limsup_n H_n = \{y \in X : \exists n_k \uparrow +\infty, n_k \in N, \exists y_{n_k} \in H_{n_k}, k \in N, \text{ with } \lim_k y_{n_k} = y\}.$$

Definition 2.5 [20] A set-valued mapping F from a topological space (W, τ) to a topological space (Z, σ) is called

(i) (τ, σ)-closed if for every $x \in K$, for every sequence $\{x_n\}$ τ-converging to x, and for every sequence $\{y_n\}$ σ-converging to a point y, such that $y_n \in F(x_n)$, one has $y \in F(x)$, i.e.,

$$F(x) \supseteq \limsup_n F(x_n).$$

(ii) (τ, σ)-lower semicontinuous if for every $x \in K$, for every sequence $\{x_n\}$ τ-converging to x, and for every $y \in F(x)$, there exists a sequence $\{y_n\}$ σ-converging to y, such that $y_n \in F(x_n)$ for n sufficiently large, i.e.,

$$F(x) \subset \liminf_n F(x_n).$$

(iii) (τ, σ)-subcontinuous on K, if for every sequence $\{x_n\}$ τ-converging in K, every sequence $\{y_n\}$, such that $y_n \in F(x_n)$, has a σ-convergent subsequence.

Definition 2.6 [20] Let A be a nonempty subset of X. The measure of non-compactness μ of the set A is defined by

$$\mu(A) = \inf\left\{\varepsilon > 0 : A \subseteq \bigcup_{i=1}^{n} A_i, \quad \mathrm{diam} A_i < \varepsilon, \quad i = 1, 2, \ldots, n\right\},$$

where diam means the diameter of a set.

Definition 2.7 [20] Let (X, d) be a metric space and let A, B be nonempty subsets of X. The Hausdorff distance $H(\cdot, \cdot)$ between A and B is defined by

$$H(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$. Let $\{A_n\}$ be a sequence of nonempty subsets of X. We say that A_n converges to A in the sense of Hausdorff distance if $H(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic, we refer the readers to [20].

Now, we prove the following lemma.

Lemma 2.1 *Suppose that set-valued mappings S and T are nonempty convex-valued, the function $f(\cdot, y)$ is convex on C for any $y \in D$, and the function $g(x, \cdot)$ is convex on D for any $x \in C$. Then $(x_0, y_0) \in \Gamma$ if and only if the following two conditions hold:*

$$x_0 \in S(x_0, y_0), \ f(x_0, y_0) \le f(z, y_0) + \frac{\alpha}{2}\|x_0 - z\|^2, \quad \forall z \in S(x_0, y_0), \tag{2.1}$$

$$y_0 \in T(x_0, y_0), \ g(x_0, y_0) \le g(x_0, w) + \frac{\alpha}{2}\|y_0 - w\|^2, \quad \forall w \in T(x_0, y_0). \tag{2.2}$$

Proof The necessity is obvious. For the sufficiency, suppose that (2.1) and (2.2) hold. Now we deduce that $(x_0, y_0) \in \Gamma$. In fact, let $z_1 \in S(x_0, y_0)$ and for any $t \in [0, 1]$, $z_t = tz_1 + (1-t)x_0$. Since $S(x_0, y_0)$ is convex, $z_t \in S(x_0, y_0)$ and so

$$f(x_0, y_0) \le f(z_t, y_0) + \frac{\alpha}{2}\|x_0 - z_t\|^2, \quad \forall t \in (0, 1].$$

By the convexity of $f(\cdot, y)$ for any $y \in D$,

$$f(x_0, y_0) \le tf(z_1, y_0) + (1-t)f(x_0, y_0) + \frac{\alpha}{2}t^2\|x_0 - z_1\|^2, \quad \forall t \in (0, 1],$$

which implies that

$$tf(x_0, y_0) \leq tf(z_1, y_0) + \frac{\alpha}{2}t^2\|x_0 - z_1\|^2, \quad \forall t \in (0, 1].$$

Thus, dividing by t in above inequality, we have

$$f(x_0, y_0) \leq f(z_1, y_0) + \frac{\alpha}{2}t\|x_0 - z_1\|^2, \quad \forall t \in (0, 1], \quad \forall z_1 \in S(x_0, y_0). \tag{2.3}$$

By the similar arguments,

$$g(x_0, y_0) \leq g(x_0, w_1) + \frac{\alpha}{2}t\|y_0 - w_1\|^2, \quad \forall t \in (0, 1], \quad \forall w_1 \in T(x_0, y_0). \tag{2.4}$$

The combination of (2.3) and (2.4) implies, for t converging to zero, that (x_0, y_0) is a solution of (SQEP). This completes the proof. □

3 Metric characterizations of α-well-posedness for (SQEP)

In this section, we shall investigate some metric characterizations of α-well-posedness for (SQEP) defined in section 2.

For any $\varepsilon > 0$, the α-approximating solution set of (SQEP) is defined by

$$M_\varepsilon = \left\{ \begin{array}{c} (x_0, y_0) \\ \in C \times D \end{array} \middle| \begin{array}{l} x_0 \in B(S(x_0, y_0), \varepsilon),\ f(x_0, y_0) - f(z, y_0) \leq \varepsilon + \frac{\alpha}{2}\|x_0 - z\|^2,\ \forall z \in S(x_0, y_0) \\ y_0 \in B(T(x_0, y_0), \varepsilon),\ g(x_0, y_0) - g(x_0, w) \leq \varepsilon + \frac{\alpha}{2}\|y_0 - w\|^2,\ \forall w \in T(x_0, y_0) \end{array} \right\}.$$

Theorem 3.1 *SQEP is α-well-posed if and only if the solution set Γ of (SQEP) is nonempty and*

$$\lim_{\varepsilon \to 0} \operatorname{diam} M_\varepsilon = 0. \tag{3.1}$$

Proof Suppose that (SQEP) is α-well-posed. Then, Γ is a singleton point set, and $M_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$, since $\Gamma \subset M_\varepsilon$. Suppose by contradiction that

$$\lim_{\varepsilon \to 0} \operatorname{diam} M_\varepsilon > \beta > 0.$$

Then there exists $\varepsilon_n > 0$ with $\varepsilon_n \to 0$, and $(w_n, z_n), (\overline{w}_n, \overline{z}_n) \in M_{\varepsilon_n}$ such that

$$\|(w_n, z_n) - (\overline{w}_n, \overline{z}_n)\| > \beta, \quad \forall n \in N.$$

Since $(w_n, z_n), (\overline{w}_n, \overline{z}_n) \in M_{\varepsilon_n}$, and (SQEP) is α-well-posed, the sequence $\{(w_n, z_n)\}$ and $\{(\overline{w}_n, \overline{z}_n)\}$, which are both α-approximating sequences for (SQEP), strongly converge to the unique solution (x_0, y_0), and this gives a contradiction. Therefore, (3.1) holds.

Conversely, let (3.1) hold and $\{(x_n, y_n)\}$ be an α-approximating sequence for (SQEP). Then, there exists a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$d(x_n, S(x_n, y_n)) \leq \varepsilon_n, \text{ and } f(x_n, y_n) - f(z, y_n) \leq \varepsilon_n + \frac{\alpha}{2}\|x_n - z\|^2, \quad \forall z \in S(x_n, y_n),$$

$$d(y_n, T(x_n, y_n)) \leq \varepsilon_n, \text{ and } g(x_n, y_n) - g(x_n, w) \leq \varepsilon_n + \frac{\alpha}{2}\|y_n - w\|^2, \quad \forall w \in T(x_n, y_n).$$

This implies that $\{(x_n, y_n)\} \subset M_{\varepsilon_n}, \forall n \in N$. Let (x_0, y_0) be the unique solution of (SQEP). Note that $(x_0, y_0) \in M_{\varepsilon_n}$. This fact together with (3.1) yields

$$\|(x_n, y_n) - (x_0, y_0)\| \le \text{diam} M_{\varepsilon_n} \to 0.$$

Thus, (SQEP) is α-well-posed. This completes of the proof. □

Theorem 3.2 *Assume that the following conditions hold:*

(i) *set-valued mappings S and T are nonempty convex-valued, (s, w)-closed, (s, s)-lower semicontinuous and (s, w)-subcontinuous on $C \times D$;*
(ii) *functions f and g are continuous on $C \times D$;*
(iii) *for any $y \in D$, the function $f(\cdot, y)$ is convex on C; for any $x \in C$, the function $g(x, \cdot)$ is convex on D.*

Then, (SQEP) is α-well-posed if and only if

$$M_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \text{ and } \lim_{\varepsilon \to 0} \text{diam } M_\varepsilon = 0. \tag{3.2}$$

Proof The necessity has been proved in Theorem 3.1. For the sufficiency, let condition (3.2) hold. It is easy to see that condition (3.2) implies that Γ is a singleton point set. Let $\{(x_n, y_n)\} \subset C \times D$ be α-approximating sequence for (SQEP). Then there exists a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$d(x_n, S(x_n, y_n)) \le \varepsilon_n, \text{ and } f(x_n, y_n) - f(z, y_n) \le \varepsilon_n + \frac{\alpha}{2}\|x_n - z\|^2, \quad \forall z \in S(x_n, y_n),$$

$$d(y_n, T(x_n, y_n)) \le \varepsilon_n, \text{ and } g(x_n, y_n) - g(x_n, w) \le \varepsilon_n + \frac{\alpha}{2}\|y_n - w\|^2, \quad \forall w \in T(x_n, y_n).$$

This means $\{(x_n, y_n)\} \subset M_{\varepsilon_n}, \forall n \in N$. It follows from (3.2) that $\{(x_n, y_n)\}$ is a Cauchy sequence and converges to a point $(x_0, y_0) \in C \times D$. In order to obtain that (x_0, y_0) solves (SQEP), we start to prove that

$$d(x_0, S(x_0, y_0)) \le \liminf_n d(x_n, S(x_n, y_n)) \le \lim_n \varepsilon_n = 0.$$

Indeed, suppose that the left inequality dose not hold. Then there exists a positive number γ such that

$$\liminf_n d(x_n, S(x_n, y_n)) < \gamma < d(x_0, S(x_0, y_0)),$$

or equivalently, there exist an increasing sequence $\{n_k\}$ and a sequence $\{z_k\}$, $z_k \in S(x_{n_k}, y_{n_k})$, $\forall k \in N$ such that

$$\|x_{n_k} - z_k\| < \gamma, \quad \forall k \in N.$$

Since the set-valued mapping S is (s, w)-closed and (s, w)-subcontinuous, the sequence $\{z_k\}$ has a subsequence, still denoted $\{z_k\}$, weakly converging to a point $z_0 \in S(x_0, y_0)$. It follows that

$$\gamma < d(x_0, S(x_0, y_0)) \le \|x_0 - z_0\| \le \liminf_k \|x_{n_k} - z_k\| < \gamma.$$

We obtain a contradiction. Thus $x_0 \in S(x_0, y_0)$. Similarly, we can prove $y_0 \in T(x_0, y_0)$.

To complete the proof, consider an arbitrary $z \in S(x_0, y_0)$. Since S is (s, s)-lower semicontinuous, there exists a sequence $\{z_n\}$ strongly converging to z, such that $z_n \in S(x_n, y_n)$ for n sufficiently large. It follows from condition (ii) that

$$\begin{aligned} f(x_0, y_0) &= \lim_n f(x_n, y_n) \\ &\le \lim_n (f(z_n, y_n) + \varepsilon_n + \frac{\alpha}{2}\|x_n - z_n\|) \\ &= f(z, y_0) + \frac{\alpha}{2}\|x_0 - z\|, \end{aligned}$$

for all $z \in S(x_0, y_0)$. Analogously, we have

$$g(x_0, y_0) \le g(x_0, w) + \frac{\alpha}{2}\|y_0 - w\|, \quad \forall w \in T(x_0, y_0).$$

It follows from Lemma 2.1 that $(x_0, y_0) \in \Gamma$. Therefore, (SQEP) is α-well-posed. This completes of the proof. □

To illustrate Theorem 3.2, we give the following two examples.

Example 3.1 Let $X = Y = C = D = R$. Let $S(x, y) = [-|x|, |x|]$, $T(x, y) = [-|y|, |y|]$, $f(x, y) = x^2 - y^2$ and $g(x, y) = y^2 - x^2$ for all $x \in C$, $y \in D$. Obviously, the conditions (i)–(iii) of Theorem 3.2 are satisfied. Note that

$$\begin{aligned}
&\left\{(x, y) \in C \times D : d(S(x, y), x) \le \varepsilon,\ f(x, y) - f(z, y) \le \varepsilon + \frac{\alpha}{2}\|x - z\|^2, \quad \forall z \in S(x, y)\right\} \\
&= \left\{(x, y) \in C \times D : d(S(x, y), x) \le \varepsilon,\ x^2 - z^2 \le \varepsilon + \frac{\alpha}{2}(x - z)^2, \quad \forall z \in S(x, y)\right\} \\
&= \left\{(x, y) \in C \times D : d(S(x, y), x) \le \varepsilon,\ -(2+\alpha)\left(z - \frac{\alpha x}{2+\alpha}\right)^2 + \frac{4}{2+\alpha}x^2 - 2\varepsilon \le 0, \quad \forall z \in S(x, y)\right\} \\
&= \left[-\sqrt{\frac{(2+\alpha)\varepsilon}{2}}, \sqrt{\frac{(2+\alpha)\varepsilon}{2}}\right] \times R
\end{aligned}$$

and

$$\begin{aligned}
&\left\{(x, y) \in C \times D : d(T(x, y), y) \le \varepsilon,\ g(x, y) - g(x, w) \le \varepsilon + \frac{\alpha}{2}\|y - w\|^2, \quad \forall w \in T(x, y)\right\} \\
&= \left\{(x, y) \in C \times D : d(T(x, y), y) \le \varepsilon,\ y^2 - w^2 \le \varepsilon + \frac{\alpha}{2}(y - w)^2, \quad \forall w \in T(x, y)\right\} \\
&= \left\{(x, y) \in C \times D : d(T(x, y), y) \le \varepsilon,\ -(2+\alpha)\left(w - \frac{\alpha y}{2+\alpha}\right)^2 + \frac{4}{2+\alpha}y^2 - 2\varepsilon \le 0, \quad \forall w \in T(x, y)\right\} \\
&= R \times \left[-\sqrt{\frac{(2+\alpha)\varepsilon}{2}}, \sqrt{\frac{(2+\alpha)\varepsilon}{2}}\right].
\end{aligned}$$

It follows that

$$M_\varepsilon = \left[-\sqrt{\frac{(2+\alpha)\varepsilon}{2}}, \sqrt{\frac{(2+\alpha)\varepsilon}{2}}\right] \times \left[-\sqrt{\frac{(2+\alpha)\varepsilon}{2}}, \sqrt{\frac{(2+\alpha)\varepsilon}{2}}\right]$$

and so $\operatorname{diam} M_\varepsilon \to 0$ as $\varepsilon \to 0$. By Theorem 3.2, (SQEP) is α-well-posed.

Example 3.2 Let $C = D = [0, +\infty)$. Let $S(x, y) = [0, x]$, $T(x, y) = [0, y]$ and $f(x, y) = g(x, y) = -xy$ for all $x \in C$, $y \in D$. It is easy to see that the conditions (i)–(iii) of Theorem 3.2 are satisfied, and $M_\varepsilon = [0, +\infty) \times [0, +\infty)$. But, (SQEP) is not α-well-posed, since $\operatorname{diam} M_\varepsilon \nrightarrow 0$ as $\varepsilon \to 0$.

When $\alpha = 0$, we have the following result.

Corollary 3.1 *Assume that the following conditions hold:*

(i) *set-valued mappings S and T are nonempty convex-valued, (s, w)-closed, (s, s)-lower semicontinuous and (s, w)-subcontinuous on $C \times D$;*
(ii) *functions f and g are continuous on $C \times D$.*

Then, (SQEP) is well-posed if and only if

$$M_\varepsilon \ne \emptyset, \quad \forall \varepsilon > 0, \text{ and } \lim_{\varepsilon \to 0} \operatorname{diam} M_\varepsilon = 0.$$

The next theorem shows that under some suitable conditions, the α-well-posed of (SQEP) is equivalent to the existence and uniqueness of its solutions.

Theorem 3.3 *Let X and Y be two finite dimensional spaces. Suppose that the following conditions hold:*

(i) *set-valued mappings S and T are nonempty convex-valued, closed, lower semicontinuous and subcontinuous on $C \times D$;*
(ii) *the functions f and g are continuous on $C \times D$;*
(iii) *for any $y \in D$, the function $f(\cdot, y)$ is convex on C; for any $x \in C$, function $g(x, \cdot)$ is convex on D;*
(iv) *M_ε is nonempty bounded for some $\varepsilon > 0$.*

Then, (SQEP) is α-well-posed if and only if (SQEP) has a unique solution.

Proof The necessity of theorem is obvious. In order to prove the sufficiency, let (x_0, y_0) be the unique solution of (SQEP) and $\{(x_n, y_n)\}$ be an α-approximating sequence for (SQEP). Then there exists a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that,

$$d(x_n, S(x_n, y_n)) \le \varepsilon_n, \text{ and } f(x_n, y_n) - f(z, y_n) \le \varepsilon_n + \frac{\alpha}{2}\|x_n - z\|^2, \quad \forall z \in S(x_n, y_n),$$

$$d(y_n, T(x_n, y_n)) \le \varepsilon_n, \text{ and } g(x_n, y_n) - g(x_n, w) \le \varepsilon_n + \frac{\alpha}{2}\|y_n - w\|^2, \quad \forall w \in T(x_n, y_n).$$

which means $\{(x_n, y_n)\} \subset M_{\varepsilon_n}, \forall n \in N$. Let $\varepsilon > 0$ be such that M_ε is nonempty bounded. Then there exists $n_0 \in N$ such that $\{(x_n, y_n)\} \subset M_{\varepsilon_n} \subset M_\varepsilon$ for all $n \ge n_0$. Thus, $\{(x_n, y_n)\}$ is bounded and so the sequence $\{(x_n, y_n)\}$ has a subsequence $\{(x_{n_k}, y_{n_k})\}$ which converges to (x_1, y_1). Reasoning as in Theorem 3.2, one proves that (x_1, y_1) solves (SQEP). The uniqueness of the solution implies that $(x_0, y_0) = (x_1, y_1)$, and so the whole sequence $\{(x_n, y_n)\}$ converges to (x_0, y_0). Thus, (SQEP) is α-well-posed. This completes of the proof. □

Example 3.3 Let $C = D = [0, +\infty)$. Let $S(x, y) = [0, x]$, $T(x, y) = [0, y]$, $f(x, y) = x^2 - y^2$ and $g(x, y) = y^2 - x^2$ for all $x \in C, y \in D$. Clearly, the conditions (i)–(iv) of Theorem 3.3 are satisfied, and (SQEP) has a unique solution $(x_0, y_0) = (0, 0)$. By Theorem 3.3, (SQEP) is α-well-posed.

4 Metric characterizations of α-well-posedness in the generalized sense for (SQEP)

In this section, we derive some metric characterizations of α-well-posedness in the generalized sense for (SQEP) by considering the non-compactness of approximate solution set.

Theorem 4.1 *(SQEP) is α-well-posed in the generalized sense if and only if the solution set Γ of (SQEP) is nonempty compact and*

$$e(M_\varepsilon, \Gamma) \to 0 \quad as \quad \varepsilon \to 0. \tag{4.1}$$

Proof Suppose that (SQEP) is α-well-posed in the generalized sense. Then Γ is nonempty. To show Γ is compact, let $\{(x_n, y_n)\} \subset \Gamma$. Clearly, if $\{(x_n, y_n)\}$ is an approximation sequence of (SQEP), then it is also α-approximation sequence. Since (SQEP) is α-well-posed in the generalized sense, it contains a subsequence strongly converging to a point of Γ. Thus, Γ is compact. Now, we prove (4.1) holds. Suppose by contradiction that there exist $\gamma > 0$, $\varepsilon_n \to 0$, and $(x_n, y_n) \in M_{\varepsilon_n}$ such that

$$d((x_n, y_n), \Gamma) \ge \gamma. \tag{4.2}$$

Being $\{(x_n, y_n)\} \subset M_{\varepsilon_n}$, $\{(x_n, y_n)\}$ is an α-approximating sequence for (SQEP). Since (SQEP) is α-well-posed in the generalized sense, there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ strongly converging to some point of Γ. This contradicts (4.2) and so (4.1) holds.

To prove the converse, suppose that Γ is nonempty compact and (4.1) holds. Let $\{(x_n, y_n)\}$ be an α-approximating sequence for (SQEP). Then $\{(x_n, y_n)\} \subset M_{\varepsilon_n}$, and so $e(M_{\varepsilon_n}, \Gamma) \to 0$. This implies that there exists a sequence $\{(z_n, w_n)\} \subset \Gamma$ such that

$$d((x_n, y_n), (z_n, w_n)) \to 0.$$

Since Γ is compact, there exists a subsequence $\{(z_{n_j}, w_{n_j})\}$ of $\{(z_n, w_n)\}$ strongly converging to $(x_0, y_0) \in \Gamma$. Hence the corresponding subsequence $\{(x_{n_j}, y_{n_j})\}$ of $\{(x_n, y_n)\}$ strongly converges to (x_0, y_0). Therefore, (SQEP) is α-well-posed in the generalized sense. □

The following example shows that the compactness condition in Theorem 4.1 is essential.

Example 4.1 Let $C = D = [0, +\infty)$. Let $S(x, y) = [x, x + y]$, $T(x, y) = [y, x + y]$ and $f(x, y) = g(x, y) = xy$ for all $x \in C, y \in D$. Then $\Gamma = M_\varepsilon = [0, +\infty) \times [0, +\infty)$. It follows that $e(M_\varepsilon, \Gamma) \to 0$ as $\varepsilon \to 0$. Clearly, the diverging sequence $\{(n, n)\}_{n\in N}$ is an α-approximating sequence, but it has no convergent subsequence. Therefore, (SQEP) is not α-well-posed in the generalized sense.

Theorem 4.2 *Assume that the following conditions hold:*

(i) *set-valued mappings S and T are nonempty convex-valued, (s, w)-closed, (s, s)-lower semicontinuous and (s, w)-subcontinuous on $C \times D$;*
(ii) *functions f and g are continuous on $C \times D$;*
(iii) *for any $y \in D$, the function $f(\cdot, y)$ is convex on C; for any $x \in C$, the function $g(x, \cdot)$ is convex on D.*

Then, (SQEP) is α-well-posed in the generalized sense if and only if

$$M_\varepsilon \neq \emptyset, \quad \forall \varepsilon > 0, \text{ and } \lim_{\varepsilon\to 0} \mu(M_\varepsilon) = 0. \tag{4.3}$$

Proof Suppose that (SQEP) is α-well-posed in the generalized sense. By the same arguments as in Theorem 4.1, Γ is nonempty compact, and $e(M_\varepsilon, \Gamma) \to 0$ as $\varepsilon \to 0$. Clearly $M_\varepsilon \neq \emptyset$ for any $\varepsilon > 0$, since $\Gamma \subset M_\varepsilon$. Observe that for any $\varepsilon > 0$, we have

$$H(M_\varepsilon, \Gamma) = \max\{e(M_\varepsilon, \Gamma), e(\Gamma, M_\varepsilon)\} = e(M_\varepsilon, \Gamma).$$

Since Γ is compact, $\mu(\Gamma) = 0$ and the following relation holds (see for example [8]):

$$\mu(M_\varepsilon) \le 2H(M_\varepsilon, \Gamma) + \mu(\Gamma) = 2H(M_\varepsilon, \Gamma) = 2e(M_\varepsilon, \Gamma),$$

It follows that (4.3) holds.

Conversely, suppose that (4.3) holds. It is easy to prove that M_ε, for any $\varepsilon > 0$, is closed. Note that $M_\varepsilon \subset M_{\varepsilon'}$ whenever $\varepsilon < \varepsilon'$, their intersection M is nonempty, compact and satisfies: $\lim_{\varepsilon\to 0} H(M_\varepsilon, M) = 0$ ([20], p. 412), where

$$M = \left\{ (x_0, y_0) \in C \times D \;\middle|\; \begin{array}{l} x_0 \in S(x_0, y_0),\ f(x_0, y_0) - f(z, y_0) \le \frac{\alpha}{2}\|x_0 - z\|^2, \quad \forall z \in S(x_0, y_0) \\ y_0 \in T(x_0, y_0),\ g(x_0, y_0) - g(x_0, w) \le \frac{\alpha}{2}\|y_0 - w\|^2, \quad \forall w \in T(x_0, y_0) \end{array} \right\}.$$

By Lemma 2.1, we obtain that M coincides with solution set Γ of (SQEP). Thus, Γ is compact.

Let $\{(x_n, y_n)\}$ be an α-approximating sequence for (SQEP). Then there exists a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$d(x_n, S(x_n, y_n)) \le \varepsilon_n, \text{ and } f(x_n, y_n) - f(z, y_n) \le \varepsilon_n + \frac{\alpha}{2}\|x_n - z\|^2, \quad \forall z \in S(x_n, y_n),$$

$$d(y_n, T(x_n, y_n)) \le \varepsilon_n, \text{ and } g(x_n, y_n) - g(x_n, w) \le \varepsilon_n + \frac{\alpha}{2}\|y_n - w\|^2, \quad \forall w \in T(x_n, y_n).$$

Thus $\{(x_n, y_n)\} \subset M_{\varepsilon_n}$. It follows from (4.3) that there exists a sequence $\{(z_n, w_n)\} \subset \Gamma$ such that

$$\|(x_n, y_n) - (z_n, w_n)\| = d((x_n, y_n), \Gamma) \le e(M_{\varepsilon_n}, \Gamma) = H(M_{\varepsilon_n}, \Gamma) \to 0.$$

Since Γ is compact, there exists a subsequence $\{(z_{n_k}, w_{n_k})\}$ of $\{(z_n, w_n)\}$ strongly converging to $(x_0, y_0) \in \Gamma$. Hence, the corresponding subsequence $\{x_{n_k}, y_{n_k}\}$ of $\{(x_n, y_n)\}$ strongly converges to (x_0, y_0). Thus, (SQEP) is α-well-posed in the generalized sense. □

Example 4.2 Let $C = D = [0, 1]$. Let $S(x, y) = [0, x]$, $T(x, y) = [0, y]$ and $f(x, y) = g(x, y) = -xy$ for all $x \in C, y \in D$. Obviously, the conditions (i)–(iii) of Theorem 4.2 are satisfied, and $M_\varepsilon = [0, 1] \times [0, 1]$. By Theorem 4.2, (SQEP) is α-well-posed in the generalized sense.

We now give a sufficient condition for the α-well-posedness in the generalized sense of (SQE) in finite dimensional spaces.

Theorem 4.3 *Let X and Y be two finite dimensional spaces. Suppose that the following conditions hold:*

- (i) *set-valued mappings S and T are nonempty convex-valued, closed, lower semicontinuous and subcontinuous on $C \times D$;*
- (ii) *functions f and g are continuous on $C \times D$;*
- (iii) *for any $y \in D$, the function $f(\cdot, y)$ is convex on C; for any $x \in C$, the function $g(x, \cdot)$ is convex on D;*
- (iv) *M_ε is nonempty bounded for some $\varepsilon > 0$.*

Then, (SQEP) is α-well-posed in the generalized sense.

Proof Let $\{(x_n, y_n)\}$ be an α-approximating sequence for (SQEP). Then there exists a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ such that

$$d(x_n, S(x_n, y_n)) \le \varepsilon_n, \text{ and } f(x_n, y_n) - f(z, y_n) \le \varepsilon_n + \frac{\alpha}{2}\|x_n - z\|^2, \quad \forall z \in S(x_n, y_n),$$

$$d(y_n, T(x_n, y_n)) \le \varepsilon_n, \text{ and } g(x_n, y_n) - g(x_n, w) \le \varepsilon_n + \frac{\alpha}{2}\|y_n - w\|^2, \quad \forall w \in T(x_n, y_n).$$

As proved in Theorem 3.3, $\{(x_n, y_n)\}$ is bounded. Then there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ which converges to (x_0, y_0). Reasoning as in Theorem 3.2, one prove that (x_0, y_0) solves (SQEP). Therefore, (SQEP) is α-well-posed in the generalized sense. □

The following example shows that the condition (iv) in Theorem 4.3 is essential.

Example 4.3 Let $C = D = [0, +\infty)$. Let $S(x, y) = [0, x]$, $T(x, y) = [0, y]$ and $f(x, y) = g(x, y) = 0$ for all $x \in C, y \in D$. It is easy to see that the conditions (i)–(iii) of Theorem 4.3 are satisfied. But $M_\varepsilon = [0, +\infty) \times [0, +\infty)$ is unbounded. Therefore, (SQEP) is not α-well-posed in the generalized sense.

5 Conclusion

In this paper, we generalize the concept of α-well-posedness to symmetric quasi-equilibrium problems which includes equilibrium problems, Nash equilibrium problems, quasivariational inequalities, variational inequalities and fixed point problems as special cases. Under some suitable conditions, we get some metric characterizations of α-well-posedness for symmetric quasi-equilibrium problems in Banach spaces. The results presented in this paper generalize and improve some known results due to Ceng et al. [5], Ceng and Yao [4], Fang et al. [10], Fang et al. [11], Lignola [23], and Lignola and Morgan [26].

(1) If $S(x, y) = C$, $T(x, y) = D$ for all $(x, y) \in C \times D$, then the α-well-posed for (SQEP) was investigated by Lignola and Morgan [26];
(2) If $X = Y$, $S(x, y) = C = D$, $T(x, y) = 0$, $g(x, y) = 0$ for all $(x, y) \in C \times D$, $x_0 = y_0$ and $f(x_0, y_0) = 0$, then the well-posed for (SQEP) was studied by Fang et al. [10];
(3) If $X = Y$, $C = D$, $S(x, y) = S(x)$, $T(x, y) = 0$, $g(x, y) = 0$ for all $(x, y) \in C \times D$, $x_0 = y_0$, and $f(x, y) = \langle u, -\eta(y, x)\rangle + h(x) - h(y)$ for all $(x, y) \in C \times C$ with some $u \in Ay$, where $\eta : C \times C \to X$ with $\eta(x, x) = 0$, $A : C \to 2^{X^*}$ and $h : C \to R$ are three mappings, and X^* denotes the dual space of X, then the well-posed for (SQEP) was considered by Ceng et al. [5];
(4) If X is a Hilbert space, $X = Y = C = D$, $S(x, y) = X$, $T(x, y) = 0$, $g(x, y) = 0$ for all $(x, y) \in C \times D$, $x_0 = y_0$, and $f(x, y) = \langle Fu, x - y\rangle + h(x) - h(y)$ for all $(x, y) \in X \times X$ with some $u \in Ay$, where $F : X \to X$, $A : X \to 2^X$ and $h : X \to R$ are three mappings, then the well-posed for (SQEP) was investigated by Ceng and Yao [4];
(5) If X is a Hilbert space, $X = Y = C = D$, $S(x, y) = X$, $T(x, y) = 0$, $g(x, y) = 0$ for all $(x, y) \in C \times D$, $x_0 = y_0$, and $f(x, y) = \langle Fy, x - y\rangle + h(x) - h(y)$ for all $(x, y) \in X \times X$, where $F : X \to X$ and $h : X \to R$ are two mappings, then the well-posed for (SQEP) was studied by Fang et al. [11];
(6) If $X = Y$, $C = D$, $S(x, y) = S(x)$, $T(x, y) = 0$, $g(x, y) = 0$ for all $(x, y) \in C \times D$, $x_0 = y_0$, and $f(x, y) = \langle Fy, y - x\rangle$ for all $(x, y) \in X \times X$, where $F : X \to X^*$ and X^* denotes the dual space of X, then the well-posed for (SQEP) was considered by Lignola [23].

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